

“Formulas for Fair, Reasonable and Non-Discriminatory Royalty Determination”

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August, 2007

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**ABSTRACT**

This paper takes an axiomatic approach to determining “Fair, Reasonable, and Non-Discriminatory” (“FRAND”) royalties for intellectual property (“IP”) rights. Drawing on the extensive game theory literature on “surplus sharing/cost sharing” problems, I describe specific formulas for determining license fees that can be derived from basic fairness principles. In particular, I describe the Shapley Value, the Proportional Sharing Rule and the Nucleolus. The Proportional Sharing Rule has the advantage that it is the only rule that is invariant to mergers and splitting of the IP owners. I also explain why, at times, there may be no acceptable to solution. Further, I contrast these rules with the Efficient Component Pricing Rule (“ECPR”) suggested by Baumol and Swanson. Unlike, the ECPR, the rules identified in this paper can uniquely determine license fees when there is more than one owner of essential IP, and also incorporate various notions of fairness and equity.

## I. INTRODUCTION

Owners of intellectual property (“IP”) are asked or required to agree to Reasonable and Non-Discriminatory or Fair Reasonable and Non-Discriminatory (“FRAND”) royalty rates as a condition for inclusion of their IP in a standard. I provide a formulaic interpretation of different FRAND principles or axioms. This paper informs policy decisions in at least three ways: (1) by providing specific formulas, each based on a set of fairness axioms, (2) showing how the solution for determining FRAND royalty rates depends on the specific axioms chosen and (3) explaining the implications of policy decisions setting royalty rates or to impose specific fairness principles in how royalty rates should be set.

The particular aim of the paper is to provide specific formulas based on fairness axioms for dividing the surplus arising from an allocation of rights to multiple patents among the patents’ owners. Most of what follows applies these principles applied to a situation in which there are at least two IP owners, and one or more licensees. However, the same principles can be applied to the case in which there is only one upstream IP owner. I present a number of different specific royalty sharing arrangements each based on a specific set of fairness axioms. This paper does not address the related issue of what are profit-maximizing royalty rates for the IP owners to charge, nor how a fair allocation of royalties is to be realized.

The issue of how to determine FRAND royalties has been addressed in a number of recent papers. These recent papers have tended to focus on a single solution. However, as explained below, the game theory literature shows that there can be many different solutions, or no solutions, to what is *fair* depending on what fairness principles

are assumed. In particular, Baumol and Swanson (2005) have suggested applying the “Efficient Component Pricing Rule” or “ECPR” to the determination of the royalty rates. Baumol and Swanson argue that ECPR will result in efficient allocations and assignment of IP rights. Layne-Farrar, Padilla and Schmalensee (2006) have suggested a different rule, the Shapley Value. The ECPR and the Shapley Value are but two of a class of solutions to what is called “surplus sharing games.”

More specifically, a surplus sharing game is a set of “players”, that is, the participants in the negotiation process to set royalty rates, a function describing the surplus that each subset of players, as well as the coalition of the whole, can create. In other words, a surplus sharing game is a description of the total profits available, as well as the profits any subset of players can assure themselves on their own. A “solution” to a surplus sharing game is a rule for allocating surplus among the players.

The Shapley Value is one such solution. In particular, the Shapley Value allocates a unique set of payoffs, or an essentially unique set of royalty rates. The ECPR is another, partial, solution, in that specifies a set of royalty rates that satisfy one efficiency axiom. This paper describes other sets of fairness axioms that can be applied to determine FRAND royalties. I show how different sets of axioms will result in different algebraic formulas for determining royalty rates.

I also show how the allocation of surplus can be quite sensitive to the particular fairness axioms being applied. I provide a number of examples showing that, at times, no allocation can satisfy all the desired fairness requirements. This means analysis that conflicts can arise between different sets of fairness criteria. In such cases, a decision about what constitutes FRAND royalty rates necessarily involves a decision about which

fairness criteria matter more than others. There may be no solution to the problem of determining FRAND royalties if the solution must satisfy too many axioms.

This paper relies heavily on two survey papers, one by H. P. Young (1994) on cost allocation and the other by H. Moulin (2002) providing an axiomatic analysis of cost and surplus sharing ‘games’. Most of the economics literature focuses on the cost allocation problems, that is, how a joint and common cost of a project should be fairly allocated among the set of beneficiaries of the project. However, as Moulin explains, much of the analysis of cost allocation problem applies to surplus sharing problems. In contrast, the allocation problem of a surplus sharing game is to fairly divide the benefits of a project.

## **II. SURPLUS SHARING GAMES**

### **a. The Basic Surplus Sharing Problem.**

When several individual entities contribute varying amounts to a project or technology, there will be different possible approaches to dividing the surplus. Each entity or “agent” will contribute some amount to the project or technology. The contribution of each agent can be measured in different ways, such as the costs incurred, the number of patents, the number of essential patents or perhaps some measure of stand-alone values (which can be difficult to compute) of each agent’s contribution.

Allocation of surplus shares can affect prices, which, in turn, can affect the efficiency and total surplus. The ECPR rule focuses on efficiency. In particular, it awards residual surplus to the owner of the IP. More specifically, the license fee is equal to the monopoly profits that the IP owner can extract. This means that another firm will

only be awarded a license when it can extract greater value or contribute supplemental value. The ECPR has an indeterminate solution when there are two or more owners of essential IP, with no separate stand-alone values. Whether the IP owners can achieve an efficient and/or joint profit maximizing solution is not the focus of the analysis. Rather, I assume that the surplus to be allocated is fixed, at least for each coalition, including the coalition of all IP owners. Further, I assume that the surplus is only being shared among IP owners, and that consumer surplus does not enter into the calculations.<sup>1</sup>

The following provides the formal mathematical expression of a surplus sharing problem. In what follows, I let  $\mathcal{N} = \{1, 2, \dots, N\}$  be the set of agents (patent holders). For any subset  $K \subseteq \mathcal{N}$ , I let  $V(K)$  denote the surplus that coalition  $K$  can guarantee itself. I also let  $V^{\mathcal{N}}$  denote the total value or surplus to be shared. A *surplus sharing game* is a pair  $(\mathcal{N}, V)$ .

At times, it can be useful to measure surplus relative to investments or number of patents. I let  $\mathbf{x} = (x_1, x_2, \dots, x_N)$  denote the share of costs, or number of patents, or another measure of value, attributable to each agent. In such case, it will be assumed that  $V^{\mathcal{N}} \geq x^{\mathcal{N}}$ , that is, there is some surplus to be shared, where  $z^{\mathcal{N}} \equiv \sum z_j$  for any  $\mathbf{z} = (z_1, z_2, \dots, z_N)$

A *solution* to a surplus sharing game is a mapping  $\phi(\mathcal{N}, V)$ , where  $\phi_j(\mathcal{N}, V)$  is agent  $j$ 's share of the total surplus. This last condition requires that the entire surplus is allocated.

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<sup>1</sup> I also do not consider the problem of the division of the surplus between the IP owners and the licensees.

## **b. Basic Fairness Concepts and Why License Fees Cannot Always be Fair**

Here I present some assumptions that have been addressed in previous papers. I then provide a simple example that illustrates that even relatively simple fairness requirements cannot always be met in practice.

One criterion of reasonableness that is especially prominent is the notion of *efficiency*, or *Pareto Efficiency*. An efficient allocation is one for which it is not possible to find a reallocation that improves the welfare of one agent without reducing the welfare on any other agent. In the context of surplus sharing games, this means that an efficient solution should allocate the entire surplus.

A second axiom that has a prominent role in economic analysis of fairness is that the solution should not provide any agent or coalition less than that agent or coalition can gain on its own. Allocations that satisfy this condition are called *core* allocations. Core allocations are also efficient, and are also called the Pareto Efficient set.

In some situations, core allocations fail to exist. Consider the following example in which there are three patents, owned by three separate entities, A, B and C. Suppose, further, that the value of any one is 1, the value of any pair of patents is 8 and the value of all three is 10. In this case, each patent owner must receive 1, any pair of patent owners must receive 8 and all three must receive 10. Suppose  $a$ ,  $b$  and  $c$  denotes the allocation of the surplus to the three patent owners.

The first set of conditions requires  $a, b$  and  $c \geq 1$ . The second set of conditions require  $(a + b) \geq 8$ ,  $(a + c) \geq 8$  and  $(b + c) \geq 8$ . The third condition requires  $(a + b + c) = 10$ . The sum of the first three inequalities is  $2(a + b + c) \geq 24$  or  $(a + b + c) \geq 12$ . This is inconsistent with the requirements that  $(a + b + c) = 10$ . Therefore, the core is empty.

This situation has the property that any sub-coalition can realize a larger than proportional share of the surplus available to all the firms together. Whenever this is the case, the core will be empty.

The requirement that an allocation of royalties be in the core does not address the split of surplus, other than requiring that no individual agent or group of agents receive no less than each can guarantee for itself. A number of other axioms can be imposed to ensure a fair and reasonable split. Lloyd Shapley suggested one set of axioms, which has the advantage that an exact, calculable expression can be shown to be the only assignment of royalties that satisfy these shares. Shapley's axioms are (See Shapley (1953) and Young (1994)):

- (1) **Additive sharing rule:** A sharing rule  $\varphi$  is *additive* if for any two "value functions"  $V'$  and  $V''$ ,  $\varphi(\mathcal{N}, V') + \varphi(\mathcal{N}, V'') = \varphi(\mathcal{N}, V' + V'')$ . In other words, if one were to add, agent by agent, the surplus shares from two separate regions or sets of licensees, the outcome would be the same as if the regions or sets of licensees had been combined.
- (2) **Symmetry:** The sharing rule  $\varphi$  is *symmetric* if the rule is invariant under any renaming of the agents or IP owners. More formally, consider two games,  $(\mathcal{N}, V)$  and  $(\mathcal{N}, W)$ . Further, suppose  $W$  is a permutation,  $\pi$ , of  $V$ . Then  $\varphi(\mathcal{N}, W) = \pi \circ \varphi(\mathcal{N}, V)$ .
- (3) **Allocates nothing to dummies:** An agent,  $j$ , is a *dummy* when  $V(S \cup \{j\}) = V(S)$  for all  $S \subseteq \mathcal{N}$ .  $\varphi_j(\mathcal{N}, V) = 0$  whenever  $j$  is a dummy. This axiom means that

parties who do not contribute to the total surplus available should receive no share of the aggregate royalties.

**(4) Efficiency.** The entire surplus is allocated, i.e.,  $\sum \phi_j(\mathcal{N}, V) = V(N)$ .

Shapley established the following:

*For each fixed  $\mathcal{N}$ , there is a unique allocation rule  $\phi(\mathcal{N}, V)$ , defined for all value functions  $V$ , that is symmetric, charges dummies nothing and is additive, namely the Shapley Value:*

$$\phi_i(\mathcal{N}, V) = \sum_{S \subseteq N \setminus \{j\}} \frac{|S|!(|N-S|-1)!}{|N|!} [V(S \cup \{j\}) - V(S)]$$

There are several ways to interpret the Shapely value. The terms  $[V(S \cup \{j\}) - V(S)]$  describe  $j$ 's incremental contribution to any subset  $S \subseteq N$ . Therefore, the Shapley Value for an agent  $j$  represents a weighted average of what agent  $j$  contributes to coalitions that could contain  $j$ .

The following provides an example of how the Shapley Value allocates surplus. It is an asymmetric example with three IP owners.

Parties A and B contribute equally to the surplus. Party C's IP is more essential.

In particular, I assume

$$V(\emptyset) = 0$$

$$V(A) = V(B) = 20$$

$$V(C) = 40$$

$$V(A \cup B) = 50$$

$$V(A \cup C) = V(B \cup C) = 80$$

$$V(A \cup B \cup C) = 100$$

In this example,  $\phi_A(\{A,B,C\}, V) = \phi_B(\{A,B,C\}, V) = 25$  and  $\phi_C(\{A,B,C\}, V) = 50$ .

In comparison, if  $V(C) = 20$  instead of 40, and  $80 \geq V(A \cup C) = V(B \cup C) = V(A \cup B) \geq 50$ , then  $\phi_A(\{A,B,C\}, V) = \phi_B(\{A,B,C\}, V) = \phi_C(\{A,B,C\}, V) = 33\frac{1}{3}$ .

So, the Shapley Value allocates surplus based on the total contribution toward that surplus. Thus, an increase in one party's individual contribution, from 20 to 40 and of its contribution to any two party coalition from 50 to or any level not exceeding 80 to 80, increases C's share from one-third of the total surplus to two thirds of the surplus.<sup>2</sup>

The Shapley Value is remarkable in that it provides an explicit formula for the **unique** division of surplus satisfying those few axioms. However, as I will explain in the next section, the allocation is quite sensitive to the specific axioms applied. In addition, application of the Shapley Value in setting royalty rates or license fees would require measures of surplus created by each coalition, that is, of each patent and set of patents. This is not an easy task in many situations.

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<sup>2</sup> Layne-Farrar, Padilla and Schmalensee (2006) explain some properties of the Shapley Value for this type of surplus sharing game.

### **III. Implications of Fairness Axioms on Surplus Shares**

The Shapley Value is in theory a numerically computable solution based on fairness principles. This makes it an appealing “solution”, although, as I have noted above, it can be difficult to estimate the underlying “value function” needed to compute the Shapley Value. The Shapley Value also has the property that agents that contributed more to the incremental value of a coalition receive a larger share of the surplus.

The Shapley Value is, unfortunately, not the only solution based on a set of fairness axioms. Here, I provide two other “solutions”, based on other fairness axioms. The problem that arises is that at times the different fairness axioms can provide conflicting recommendations. In addition, application of any of the fairness axioms relies on ability to calculate the value functions on which they rest.

#### **a. Proportional Rationing of Surplus**

One set of conditions for setting FRAND that could be of significance is how the license fees can be affected by mergers, acquisition, sub-licensing agreements and other reassignments of IP rights among the negotiating parties. There are several ways in which re-assignment of IP ownership can affect negotiations. What follows is a list of ways in which this is possible, and the implications of “an invariance” of license fees to such reassignments. In other words, I show what division of license fees would be consistent with the requirement that the share accruing to any one patent is not affected by a reassignment of that patent.

In this section, I let  $N$  denote the set of possible agents. The actual number of negotiation entities will be  $N \leq |N|$  where  $|X|$  denotes the number of elements in a set  $X$ . Mergers and splitting (or demergers) can increase or decrease the number of negotiating parties.

I let  $\mathbf{x} = (x_1, x_2, x_3, \dots, x_N)$  denote the patents held by each party. Each  $x_j$  can represent a number or share of essential patents or an agreed or asserted value of party  $j$ 's patents. I let  $T$  denote the total surplus to be split among the parties. The analysis applies equally well whether the division of the surplus is based on numbers of patents, numbers of essential patents or measures of the patent values. In what follows, I assume that  $x_j$ 's represent claims of shares of the total surplus  $T$ . I let  $r(N, T, \mathbf{x})$  denote a method of determining license fees, or a surplus sharing rule, that allocates or rations the surplus among the  $N$  licensors.<sup>3</sup>

The following lists the axioms for the division of the license fees across patents to be independent of mergers and splits.

**i. No Advantageous Reallocation (NAR)**

For all set of agents,  $N$  and subsets,  $S$ , of the set of agents, all  $T$ , and all vectors  $\mathbf{x}$  and  $\mathbf{x}'$ : If the coalition  $S$  has the same aggregate set of licenses under  $\mathbf{x}$  and  $\mathbf{x}'$ , then  $S$  receives the same share of the total surplus, i.e.,  $r_S(N, T, \mathbf{x}) = r_S(N, T, \mathbf{x}')$ .

**ii. Irrelevance of Reallocations (IR)**

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<sup>3</sup> Note, that in the above section, I assumed that surplus sharing rule had, as arguments, a value function,  $V$ , and set of players,  $N$ . Here, the rule depends on the total surplus,  $T$ , as it is assumed that surplus is transferable across agents. In addition, in this section, the surplus sharing rule is parameterized by a vector of initial (patent) holdings,  $\mathbf{x}$ .

For all set of agents  $N$  and subsets,  $S$ , of the set of agents, all  $T$ , and all vectors  $\mathbf{x}$  and  $\mathbf{x}'$ : If the patents by entities outside of  $S$ , do not affect the allocation of surplus to members of  $S$ . In particular, if all members of  $S$  have the same patents in  $\mathbf{x}$  and  $\mathbf{x}'$ , then  $r_j(N, T, \mathbf{x}) = r_j(N, T, \mathbf{x}')$  for all  $j$  in  $S$ .

### iii. Independence of Mergers and Splitting (IMS)

For all set of agents  $N$  and subsets,  $S$ , of the set of agents, all  $T$ , and all vectors  $\mathbf{x}$ , then  $r_j(N, T, \mathbf{x}) = r_j(N', T, \mathbf{x}')$  for all  $j$  in  $S$ , where  $N$  and  $N'$  differ in that  $N'$  consists of the set of agents  $N \setminus S$  and a single merged entity,  $S$ , and  $\mathbf{x}$  differs from  $\mathbf{x}'$  in that the patents of the members of  $S$  are combined in  $\mathbf{x}'$ .

### iv. Decomposition (DEC)

For any set of agents  $N$ , and partition of  $N$  into  $K$  disjoint subsets,  $N_1, N_2, \dots, N_K$ , for any  $T, \mathbf{x}$ , and  $k$ ,  $r_j(N, T, \mathbf{x}) = r_j(N_k, T_k, \mathbf{x}')$  where  $\mathbf{x}'$  consists of the patent portfolio of the parties in coalition  $N_k$ , and  $T_k$  is the surplus accruing to agents in  $N_k$  as determined by  $r(N, T, \mathbf{x})$ . In other words, the rule for determining allocation of surplus must satisfy the property that individual entities in any subset would receive the same shares when the negotiation occurs among the subset of agents, starting with the subset's share of the total patent and surplus, as they do when the negotiation occurs with an agreement among all the agents.

Some have argued that royalties should be split proportional to the number of patents. When the  $x_j$ 's *should* represent numbers of patents, rather than relative values or another measure of importance, Moulin (2002) has shown that ***a proportional allocation of surplus, that is,  $r_j(N, T, \mathbf{x}) = T \cdot x_j / \sum_k x_k$  satisfies NAR, IR, IMS and DEC, and when***

*there are three or more agents, it is the only allocation method meeting any one of the four above properties.*

### **b. The Nucleolus**

As I noted above the Shapley Value is only one allocation rule. Other axioms result in different allocation rules. Here, I describe one, the **nucleolus**.

When the core is non-empty, the nucleolus is the allocation that provides each agent the same surplus above what each can guarantee itself.<sup>4</sup> The nucleolus is the unique cost allocation that is **symmetric**, **invariant** when decomposed into direct values and joint values, and **consistent**.

A **consistent** surplus sharing rule is one which satisfies the property that it would provide any subgroup to the same division for the subgroup as a separate surplus sharing game as it provides that subgroup as part of the larger surplus sharing game.

A surplus sharing rule that is **invariant** when surplus can be decomposed into direct values and joint values is one that satisfies the following property:

Suppose, for any  $\alpha > 0$ , and  $\beta \in \mathbf{R}^N$ , and  $S \subseteq N$ ,  $W(S) = \alpha V(S) + \sum_{k \in S} \beta_k$ , then  $\varphi_j(N, W) = \alpha \varphi_j(N, V) + \beta_j$  for all  $j$ .

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<sup>4</sup> For example, consider a three party surplus sharing game, in which party A can guarantee itself at least 15, B 15 and C can guarantee itself at least 40. Also, suppose that when all three agree, they can achieve a joint surplus of 100. Also, assume two agent coalitions cannot achieve any greater surplus as a coalition than two agents can do on their own. Then, the core satisfies the condition that A and B get at least 15, C gets at least 40 and they share the entire 100. The nucleolus in this example is the surplus allocation of (25, 25, 50).

The Shapley Value, in contrast, does not require consistency. While both the Shapley Value and Nucleolus identify unique solutions, they may disagree. The following modification of the example of Section II illustrates this.<sup>5</sup>

$$V(\emptyset) = 0$$

$$V(A) = V(B) = 20$$

$$V(C) = 40$$

$$V(A \cup B) = 50$$

$$V(A \cup C) = V(B \cup C) = 80$$

$$V(A \cup B \cup C) = 100$$

In this example, the Shapley Value allocates to A, B and C, the amounts 25, 25 and 50. In contrast, as explained in the Appendix, the Nucleolus is  $(23\frac{1}{3}, 23\frac{1}{3}, 53\frac{1}{3})$ . Thus, the Nucleolus in this example gives a higher share of the surplus to the firm with a greater contribution to the total surplus.

#### **IV. Bargaining Models**

The above provides approaches to identifying cooperative solutions to surplus sharing royalty setting problems. These cooperative solutions assume that a fixed set of fairness principles or “axioms” can be imposed on those setting royalty rates. This cooperative game approach essentially means that all those involved have agreed or can be required to adhere to the criteria being used to set royalty rates.

An alternative is a non-cooperative, or bargaining, approach.

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<sup>5</sup> Tijs and Dreissen (1986) provide another example of this point for a cost allocation game. See Friedman (1986) for a more extensive discussion of the Shapley Value, Nucleolus and related solution concepts.

In a bargaining model, solutions are derived assuming fixed bargaining rules. For example, the Nash ((1950) and (1953)) bargaining solution solves a specific maximization problem, and the Rubinstein (1982) bargaining model derives a unique outcome assuming a fixed structure of negotiations. The axiomatic approach is not inconsistent with the bargaining models (Osborne and Rubinstein (1990)).

More specifically, in some cases, the outcome of a bargaining game can coincide with the solution to a cooperative surplus-sharing game. For example, Serrano (1995) defines a bargaining game which has a solution that coincides with the nucleolus.

## **V. Conclusion**

This paper examines alternative fairness principles, or axioms. I show how alternative sets of criteria can logically imply specific formulas for setting royalty rates. In particular, I identify four solution concepts: the core, which is not single-valued but imposes the “weakest” requirements; the Shapley Value; the nucleolus; and the proportional sharing rule.

The focus in this paper is to describe the nature of the solutions that can be derived based on an axiomatic approach to determining fair allocations of surplus. For at least two key solution concepts, the Shapley Value and the Nucleolus, this approach provides a formula for allocating surplus. This approach can be consistent with bargaining models<sup>6</sup>. This paper is not intended to provide an exhaustive analysis of what may or may not constitute FRAND royalties under different circumstances. Many other

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<sup>6</sup> See Osborne and Rubinstein (1990) for a description of the relation between axiomatic and strategic approaches to bargaining. Also, see Stole and Zweibel (1996a) and (1996b) for a discussion of an n-person bargaining model.

fairness concepts have been developed. I have analyzed four in some detail.<sup>7</sup> Moreover, this paper does not provide any systematic analysis of when there will be conflicts and/or inconsistencies between different sets of fairness axioms.

I adopt an axiomatic approach as the question of determining FRAND royalty rates is, to a large extent, one of determining a “fair” division of the surplus created – although all the solutions provided retain the more customary economic efficiency criteria. Straightforward social welfare or surplus maximization will not generally result in a determinant outcome. For that reason, royalty determination decision is largely a decision about equity.

In this paper I have not tried to identify specific translation of the alternative formulas for setting royalty rates. The formulas do tend to provide greater share of surplus, and a higher relative royalty, to those parties contributing a greater share of total surplus.<sup>8</sup> As a practical matter, the Moulin result about proportional allocation rules may have significance if the royalty rates are to be independent of mergers and acquisitions. Neither this result, nor any of the other results support simple patent counting allocation formulas. More specifically, the application of the Moulin result can require potentially differential weights being applied to each patent.

More generally, this paper describes the issues that need to be addressed in determining FRAND royalty rates. I have shown that what constitutes FRAND can be quite sensitive to the principles applied, and I provide some specific guidance as to how conflicts can arise between different fairness principles.

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<sup>7</sup> See Friedman (1986) for an exposition of other concepts, including the Kernel, the Bargaining Set, and the Banzhoff Power Index.

<sup>8</sup> See Layne-Farrar, Padilla and Schmalenses (2006) for a detailed discussion of the Shapley Value.

## APPENDIX

Example: Core, Shapley Value and Nucleolus

Three agents, A, B and C.

$$V(A) = 40$$

$$V(B) = V(C) = 20$$

$$V(A \cup B) = 80$$

$$V(B \cup C) = 50$$

$$V(A \cup C) = 80$$

$$V(A \cup B \cup C) = 100$$

$$V(A) - V(\emptyset) = 40$$

$$V(A \cup B) - V(B) = V(A \cup C) - V(C) = 60$$

$$V(A \cup B \cup C) - V(B \cup C) = 50$$

$$V(B) - V(\emptyset) = 20$$

$$V(A \cup B) - V(A) = 40$$

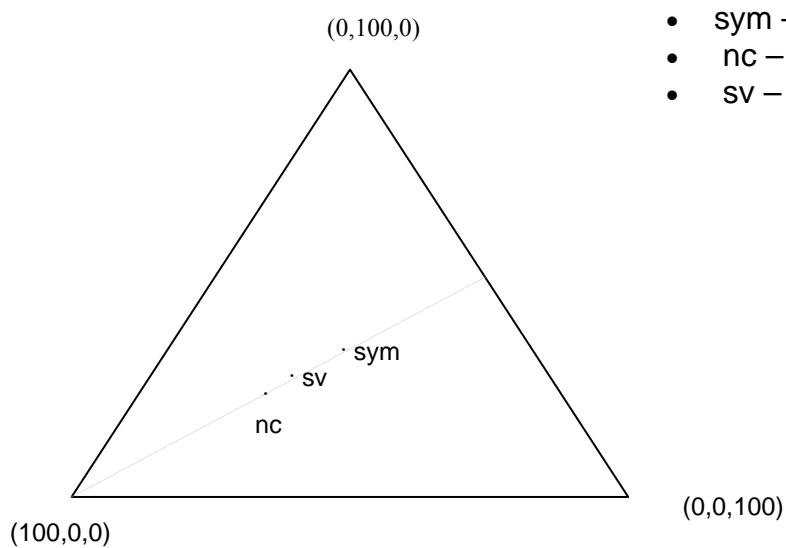
$$V(B \cup C) - V(C) = 30$$

$$V(A \cup B \cup C) - V(A \cup B) = 20$$

The let  $a$ ,  $b$  and  $c$  denote the surplus received by A, B and C, respectively.

The Core =  $\{(a, b, c) \mid a + b + c = 100, a \geq 40, b \geq 20, c \geq 20, a + b \geq 80, a + c \geq 80, \text{ and } b + c \geq 50\}$ .

The Shapley Value =  $(50, 25, 25)$ , and the Nucleolus =  $(53\frac{1}{3}, 23\frac{1}{3}, 23\frac{1}{3})$ .



- sym – Symmetric Allocation
- nc – nucleolus
- sv – Shapley Value

In the diagram, the vertices represent the entire surplus, 100, being allocated to one agent. The points on the line from the (100,0,0) bisecting the triangle represent equal allocations to agents B and C, and the amount being allocating to A varying from 100 to 0 along the line, starting at the vertex.

Example 2: Now, suppose we modify the above example, so that all three agents contribute the same surplus.

$$\begin{aligned}
 V(A) &= 20 \\
 V(B) &= V(C) = 20 \\
 V(A \cup B) &= 50 \\
 V(B \cup C) &= 50 \\
 V(A \cup C) &= 50 \\
 V(A \cup B \cup C) &= 100
 \end{aligned}$$

In this case, the Shapley Value and Nucleolus result in the same allocation of surplus –  $33\frac{1}{3}$  to each.

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